Spoon Feeding Differential Equations

Simplified Knowledge Management Classes Bangalore

My name is Subhashish Chattopadhyay. I have been teaching for IIT-JEE, Various International Exams (such as IMO [International Mathematics Olympiad], IPhO [International Physics Olympiad], IChO [International Chemistry Olympiad]), IGCSE (IB), CBSE, I.Sc, Indian State Board exams such as WB-Board, Karnataka PU-II etc since 1989. As I write this book in 2016, it is my 25 th year of teaching. I was a Visiting Professor to BARC Mankhurd, Chembur, Mumbai, Homi Bhabha Centre for Science Education (HBCSE) Physics Olympics camp BARC Campus.

I am Life Member of …

- IAPT (Indian Association of Physics Teachers)

- IPA (Indian Physics Association)

- AMTI (Association of Mathematics Teachers of India)

- National Human Rights Association

- Men's Rights Movement (India and International)

- MGTOW Movement (India and International)

And also of

IACT (Indian Association of Chemistry Teachers)

The selection for National Camp (for Official Science Olympiads - Physics, Chemistry, Biology, Astronomy) happens in the following steps ….

1) **NSEP** (National Standard Exam in Physics) and **NSEC** (National Standard Exam in Chemistry) held around 24 rth November. Approx 35,000 students appear for these exams every year. The exam fees is Rs 100 each. Since 1998 the IIT JEE toppers have been topping these exams and they get to know their rank / performance ahead of others.

2) **INPhO** (Indian National Physics Olympiad) and **INChO** (Indian National Chemistry Olympiad). Around 300 students in each subject are allowed to take these exams. Students coming from outside cities are paid fair from the Govt of India.

3) The Top 35 students of each subject are invited at HBCSE (Homi Bhabha Center for Science Education) Mankhurd, near Chembur, BARC, Mumbai. After a 2-3 weeks camp the top 5 are selected to represent India. The flight tickets and many other expenses are taken care by Govt of India.

Since last 50 years there has been no dearth of “Good Books“. Those who are interested in studies have been always doing well. This e-Book does not intend to replace any standard text book. These topics are very old and already standardized.

There are 3 kinds of Text Books

- The thin Books - Good students who want more details are not happy with these. Average students who need more examples are not happy with these. Most students who want to “Cram” quickly and pass somehow find the thin books “good” as they have to read less !!

- The Thick Books - Most students do not like these, as they want to read as less as possible. Average students are “busy” with many other things and have no time to read all these.

- The Average sized Books - Good students do not get all details in any one book. Most bad students do not want to read books of “this much thickness“ also !!

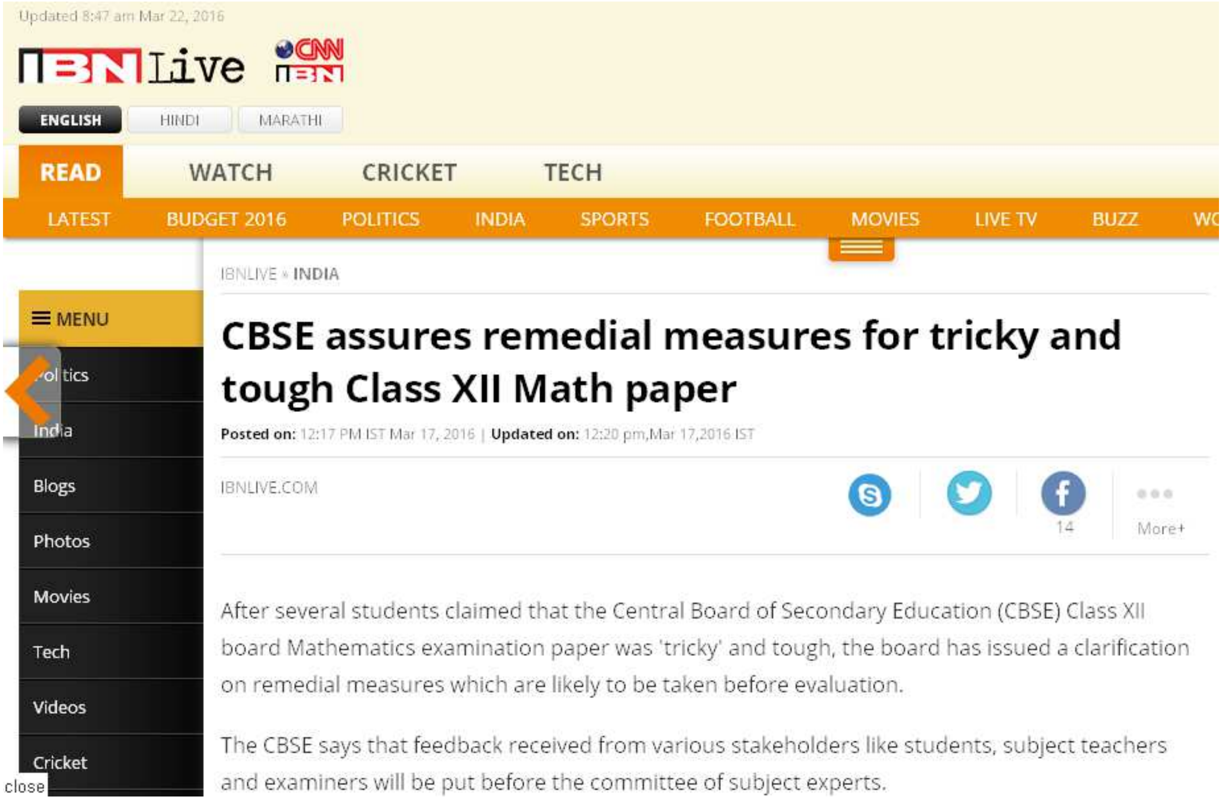
We know there can be no shoe that’s fits in all.

Printed books are not e-Books! Can’t be downloaded and kept in hard-disc for reading “later” ………

So if you read this book later, you will get all kinds of examples in a single place. This becomes a very good “Reference Material”. I sincerely wish that all find this “very useful”.

Students who do not practice lots of problems, do not do well. The rules of “doing well” had never changed …. Will never change !

After 2016 CBSE Mathematics exam lots of students complained that the paper was tough!



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CBSE assures remedial measures for tricky and tough Class XII Math paper

Posted on: 12:17 PM IST Mar 17, 2016 | **Updated on:** 12:20 pm,Mar 17,2016 IST

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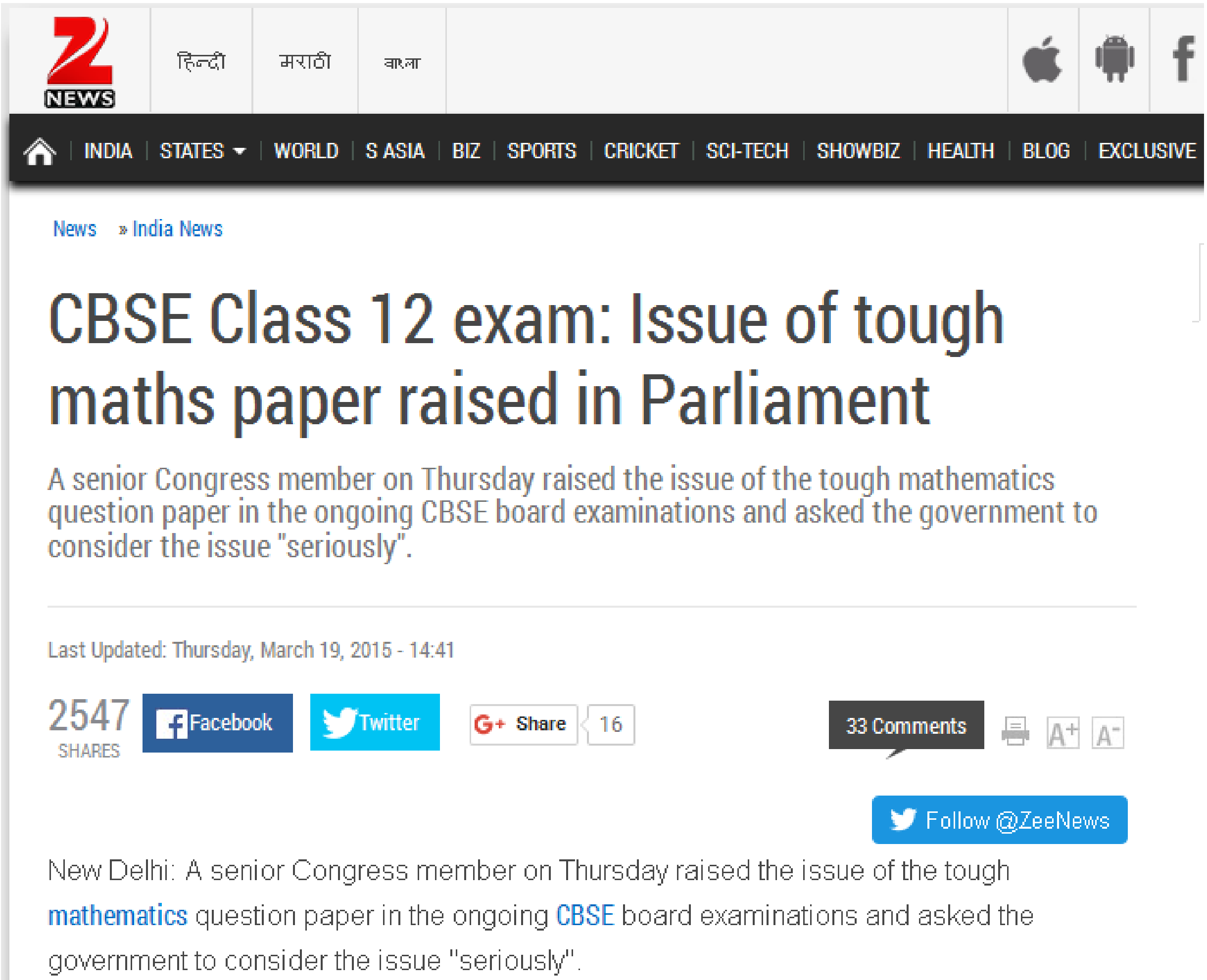
After several students claimed that the Central Board of Secondary Education (CBSE) Class XII board Mathematics examination paper was 'tricky' and tough, the board has issued a clarification on remedial measures which are likely to be taken before evaluation.

The CBSE says that feedback received from various stakeholders like students, subject teachers and examiners will be put before the committee of subject experts.

In 2015 also the same complain was there by many students



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CBSE Class 12 exam: Issue of tough maths paper raised in Parliament

A senior Congress member on Thursday raised the issue of the tough mathematics question paper in the ongoing CBSE board examinations and asked the government to consider the issue "seriously".

Last Updated: Thursday, March 19, 2015 - 14:41

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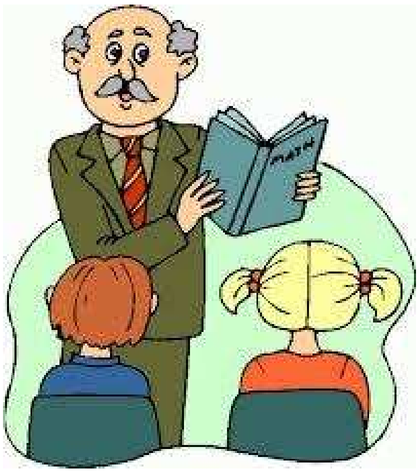
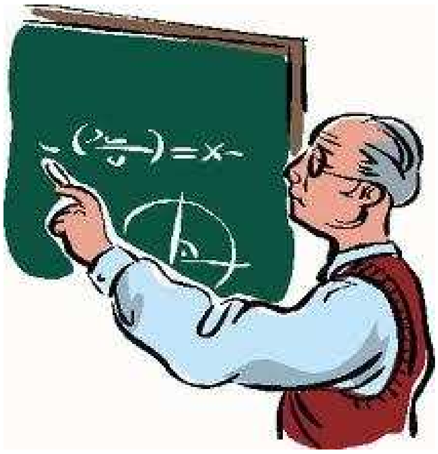
New Delhi: A senior Congress member on Thursday raised the issue of the tough mathematics question paper in the ongoing CBSE board examinations and asked the government to consider the issue "seriously".

These complains are not new. In fact since last 40 years, (since my childhood), I always see this; every year the same setback, same complain!

In this e-Book I am trying to solve this problem. Those students who practice can learn.

No one can help those who are not studying, or practicing.





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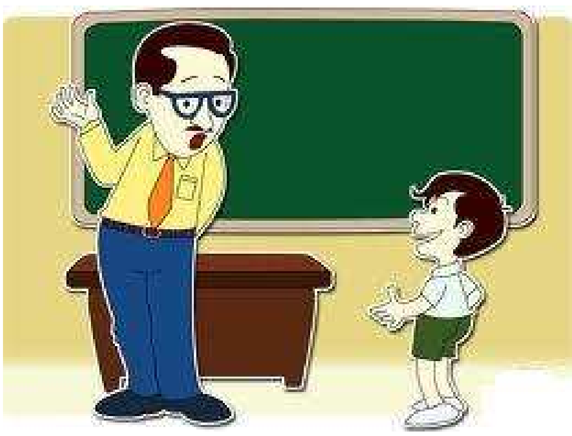

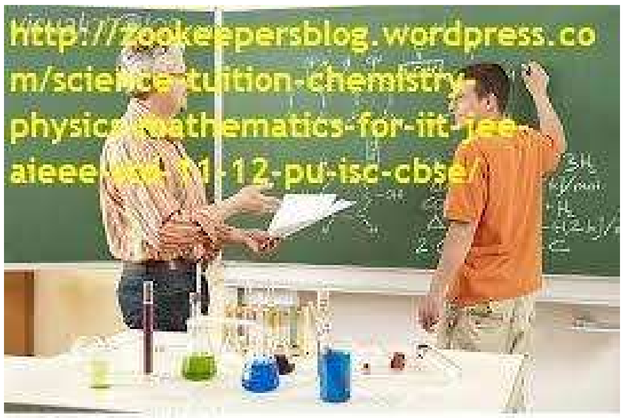

Spoon Feeding Series - Differential Equation

In any book solution techniques of various types of Differential equations will be given. But in exam when you get one, you are not sure of what type is it. So you have to try the various methods one by one ……

The approach to solve Differential Equations would be as follows.

Step –1 Check if the problem is of type variable separable

If yes then solve it

Else

Step -2 Check if it is of the type exact. This is because it is easiest or fastest to solve differential equations of exact type

Else step –3 Check if the problem is modifiable to " Exact type ". (by multiplying with a I.F (Integrating Factor))

If you could identify the multiplying factor and modified then solve it as EXACT type

Else step –4 Check if some differential coefficients can be squeezed ?

Else step –5 check if it is homogeneous type ? (or is it reducible to homogeneous) ?

Else step –6 check if it linear or modifiable to linear.

Else step –7 check if it is of the form Bernoulli (This is also modifiable to linear)

Else step –8 check if it can be written as D parameter and factorized.

But before we proceed with examples and types of Differential Equations, it is important to recall the Integration rules or methods. (This chapter assumes that the students is very good at Indefinite Integral.)

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Recall the various tricks, formulae, and rules of solving Indefinite Integrals

(i) $\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

(ii) $\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C = \frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right) + C$

(iii) $\int \frac{dx}{x^2-a^2}\,dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C = -\frac{1}{a}\coth^{-1}\left(\frac{x}{a}\right) + C$

(iv) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C$

(v) $\int \frac{dx}{\sqrt{x^2-a^2}} = \log|x+\sqrt{x^2-a^2}| + C = \cosh^{-1}\left(\frac{x}{a}\right) + C$

(vi) $\int \frac{dx}{\sqrt{x^2+a^2}} = \log|x+\sqrt{x^2+a^2}| + C = \sinh^{-1}\left(\frac{x}{a}\right) + C$

(vii) $\int \sqrt{x^2+a^2}\,dx = \frac{1}{2}\left[x\sqrt{x^2+a^2} + a^2\log|x+\sqrt{x^2+a^2}|\right] + C$

(viii) $\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}\left[x\sqrt{a^2-x^2} + a^2\sin^{-1}\left(\frac{x}{a}\right)\right] + C$

(ix) $\int \sqrt{x^2-a^2}\,dx = \frac{1}{2}\left[x\sqrt{x^2-a^2} - a^2\log|x+\sqrt{x^2-a^2}|\right] + C$

(x) $\int (px+q)\sqrt{ax^2+bx+c}\,dx = \frac{p}{2a}\int (2ax+b)\sqrt{ax^2+bx+c}\,dx$

$+\left(\frac{q-pb}{2a}\right)\int \sqrt{ax^2+bx+c}\,dx$

- $\int e^x dx = e^x$
- $\int e^{ax} dx = \frac{1}{a} e^{ax}$
- $\int e^{ax} \cos bx dx = \frac{e^{ax}}{a^2+b^2} (a \cos bx + b \sin bx)$
- $\int e^{ax} \sin bx dx = \frac{e^{ax}}{a^2+b^2} (a \sin bx - b \cos bx)$
- $\int a^x dx = \frac{a^x}{lna} + c$

- $\int \csc x \cot x dx = -\csc x + c$
- $\int \csc^2 x dx = -\cot x + c$
- $\int \sec x \tan x dx = \sec x + c$
- $\int \sec^2 x dx = \tan x + c$
- $\int \sin x dx$ = - \cos x + c$
- $\int \cos x dx = \sin x + c$

- $\int \log x dx = x(\log x - 1) + c$
- $\int \frac{1}{x} dx = \log |x| + c$
- $\int a^x dx = a^x \log x + c$
- $\int \frac{dx}{a^2-x^2} = \frac{1}{2a} \log \frac{a+x}{a-x} + c$
- $\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \log \frac{x-a}{x+a} + c$

- $\int (ax+b)^n = \frac{1}{a} \frac{(ax+b)^{n+1}}{n+1} + C$, $n \neq 1
- $\int \frac{dx}{(ax+b)} = \frac{1}{a} \log |ax+b| + C$
- $\int e^{ax+b} = \frac{1}{a} e^{ax+b} + C$
- $\int \cos(ax+b) dx = \frac{1}{a} \sin(ax+b) + C$
- $\int \sec^2 (ax+b) dx = \frac{1}{a} \tan(ax+b) + C$
- $\int \csc^2 (ax+b) dx = \frac{-1}{a} \cot(ax+b) + C$
- $\int \csc(ax+b) \cot(ax+b) dx = \frac{-1}{a} \csc(ax+b) + C$

Some advanced procedures….

$\int \frac{x^m}{(a+bx)^p}\,dx$ m is a + ve integer	Put a + bx = z
$\int \frac{dx}{x^m (a+bx)^p}$. where either ($m$ and p positive integers) or (m and p are fractions, but $m + p$ = integers > 1)	Put a + bx = zx
$\int x^m \left(a+bx^n\right)^p\,dx$, where m, n, p are rationals.	
(i) p is a + ve integer	Apply Binomial theorem to $(a + bx^n)^p$
(ii) p is a − ve integer	Put $x = z^k$ where k = common denominator of m and n.
(iii) $\frac{m+1}{n}$ is an integer	Put $(a + bx^n) = z^k$ where k = denominator of p.
(iv) $\frac{m+1}{n} + p$ is an integer	Put $a + bx^n = x^n z^k$ where k = denominator of fraction p.

Solve a Simple Problem

$$\int \frac{3x+1}{2x^2+x+1}\, dx = \int \left(\frac{\frac{3}{4}(4x+1)+\frac{1}{4}}{2x^2+x+1} \right) dx$$

$$= \frac{3}{4}\int\left(\frac{4x+1}{2x^2+x+1}\right) dx + \frac{1}{8}\int \frac{dx}{\left(x^2+\frac{x}{2}+\frac{1}{2}\right)}$$

$$= \frac{3}{4}\log(2x^2+x+1) + \frac{1}{2\sqrt{7}}\tan^{-1}\frac{4x+1}{\sqrt{7}} + C$$

-

Solve a problem

$$\int \frac{x}{(1-x)^{1/3}-(1-x)^{1/2}}dx \qquad \{\text{ The LCM of 2 and 3 is 6 }\}$$

Hence, substitute $1-x=u^6$ Then, $dx=-6u^5du$

$$\Rightarrow I = \int \frac{1-u^6}{u^2-u^3}(-6u^5du) = -6\int \frac{1-u^6}{1-u}u^3du$$

$$= -6\int (1+u+u^2+u^3+u^4+u^5)\, u^3du$$

$$= -6\left(\frac{1}{4}u^4+\frac{1}{5}u^5+\frac{1}{6}u^6+\frac{1}{7}u^7+\frac{1}{8}u^8+\frac{1}{9}u^9\right)+c$$

Solve a Problem

Evaluate $\int \cos 2x \log(1+\tan x)dx$.

Solution:

Integrating by parts taking cos 2x as the 2nd function, the given integral

$$= \{\log(1+\tan x)\}\frac{\sin 2x}{2} - \int \frac{\sec^2 x}{1+\tan x} \cdot \frac{\sin 2x}{2} dx$$

$$= \frac{1}{2}\sin 2x \log(1+\tan x) - \int \frac{\sin x}{\sin x + \cos x} dx.$$

Now $\int \frac{\sin x \, dx}{\sin x + \cos x}$

$$= \frac{1}{2}\int \frac{(\sin x + \cos x) - (\cos x - \sin x)}{\sin x + \cos x} dx,$$

$$= \frac{1}{2}\int \left[1 - \frac{\cos x - \sin x}{\sin x + \cos x}\right] dx = \frac{1}{2}[x - \log(\sin x + \cos x)].$$

Hence the given integral

$$= \frac{1}{2}\sin 2x \log(1+\tan x) - \frac{1}{2}[x - \log(\sin x + \cos x)].$$

Recall how to integrate Linear X root Quadratic in denominator

Find the value of the $\int \frac{dx}{(x+1)\sqrt{(1+2x-x^2)}}$

Putting $(x+1) = \frac{1}{t}$, so that $dx = -\frac{1}{t^2}\, dt$, $x = \frac{1-t}{t}$ and

$$(1+2x-x^2) = 1 + 2\left(\frac{1-t}{t}\right) - \frac{(1-t)^2}{t^2} = \frac{2}{t^2}\left[\left(\frac{1}{\sqrt{2}}\right)^2 - (t-1)^2\right],$$

we get the value of the given integral transformed as

$$\int \frac{-\frac{1}{t^2}\,dt}{\frac{1}{t}\frac{2}{\sqrt{t}}\left[\left(\frac{1}{\sqrt{2}}\right)^2-(t-1)^2\right]} = -\frac{1}{\sqrt{2}}\sin^{-1}\frac{t-1}{\left(\frac{1}{\sqrt{2}}\right)} + \text{C}$$

$$= \frac{1}{\sqrt{2}}\ \sin^{-1}\ \frac{\sqrt{2}\,x}{(x+1)} + C$$

Another advanced example

Example Evaluate $\int \frac{dx}{x\sqrt{(1+x^n)}}$

Make the substitution $(1 + x^n) = t^2$ or $x^n = (t^2 - 1)$, so that $n\, x^{n-1}\, dx = 2t\, dt$, we get

$$\int \frac{2t\,dt}{n x^n t} = \frac{2}{n}\int \frac{dt}{(t^2-1)} = \frac{1}{n}\ln\left|\frac{t-1}{t+1}\right|$$

$$= \frac{1}{n}\ \ln\ \left|\frac{\sqrt{(1+x^n)}-1}{\sqrt{(1+x^n)}+1}\right| + C$$

Similarly

The value of integral $\int \frac{dx}{x\sqrt{1-x^3}}$ is given by

(a) $\frac{1}{3}\log\left|\frac{\sqrt{1-x^3}+1}{\sqrt{1-x^3}-1}\right|+C$ (b) $\frac{1}{3}\log\left|\frac{\sqrt{1-x^3}-1}{\sqrt{1-x^2}+1}\right|+C$

(c) $\frac{2}{3}\log\left|\frac{1}{\sqrt{1-x^3}}\right|+C$ (d) $\frac{1}{3}\log|1-x^3|+C$

Ans. **(b)**

Solution Put $1-x^3=t^2$. Then $-3x^2dx=2t\,dt$ and the integral becomes

$$-\frac{1}{3}\int\frac{-3x^2dx}{x^3\sqrt{1-x^3}}=-\frac{1}{3}\int\frac{2t\,dt}{(1-t^2)t}=\frac{2}{3}\int\frac{dt}{t^2-1}$$

$$=\frac{2}{3}\left(\frac{1}{2}\log\left|\frac{t-1}{t+1}\right|\right)+C=\frac{1}{3}\log\left|\frac{\sqrt{1-x^3}-1}{\sqrt{1-x^3}+1}\right|+C$$

Solve a Problem

$\int\sqrt{\sec x-1}\;dx$ is equal to

(a) $2\log\left(\cos\frac{x}{2}+\sqrt{\cos^2\frac{x}{2}-\frac{1}{2}}\right)+C$

(b) $\log\left(\cos\frac{x}{2}+\sqrt{\cos^2\frac{x}{2}-\frac{1}{2}}\right)+C$

(c) $-2\log\left(\cos\frac{x}{2}+\sqrt{\cos^2\frac{x}{2}-\frac{1}{2}}\right)+C$

(d) none of these

(c). $\int \sqrt{\sec x - 1}\ dx = \int \sqrt{\frac{1-\cos x}{\cos x}}\ dx$

$$= \sqrt{2} \int \frac{\sin \frac{x}{2}}{\sqrt{2\cos^2 \frac{x}{2} - 1}}\ dx = -2\sqrt{2} \int \frac{dz}{\sqrt{2z^2 - 1}}$$

$$\left(\text{Putting } \cos \frac{x}{2} = z \Rightarrow \sin \frac{x}{2}\ dx = -2dz\right)$$

$$= -2 \int \frac{dz}{\sqrt{z^2 - \left(\frac{1}{\sqrt{2}}\right)^2}}$$

$$= -2 \log \left[z + \sqrt{z^2 - \left(\frac{1}{\sqrt{2}}\right)^2} \right] + C$$

$$= -2 \log \left(\cos \frac{x}{2} + \sqrt{\cos^2 \frac{x}{2} - \frac{1}{2}} \right) + C$$

Solve another problem

$$I = \int \sqrt{1+\operatorname{cosec} x}\ .\ dx$$

$$= \int \sqrt{1+\frac{1}{\sin x}}\ .\ dx = \int \sqrt{\frac{\sin x + 1}{\sin x}}\ .\ dx$$

$$= \int \sqrt{\frac{(1+\sin x)(1-\sin x)}{\sin x\,(1-\sin x)}}\ .\ dx \qquad \text{[On rationalization]}$$

$$= \int \sqrt{\frac{1-\sin^2 x}{\sin x - \sin^2 x}}\ .\ dx \qquad [\because\ (a+b)(a-b) = a^2 - b^2]$$

$$= \int \frac{\cos x}{\sqrt{\sin x - \sin^2 x}}\ .\ dx \qquad [\because\ \sin^2 A + \cos^2 A = 1]$$

$$\sin x = z \quad \Rightarrow \quad \cos x\, dx = dz$$

$$I = \int \frac{1}{\sqrt{z - z^2}}\ .\ dz = \int \frac{1}{\sqrt{-(z^2 - z)}}\ .\ dz$$

$$= \int \frac{1}{\sqrt{\frac{1}{4} - \left(z^2 - z + \frac{1}{4}\right)}}\ .\ dz \qquad \left[\begin{array}{l}\text{Add and subtract } \frac{1}{4} \text{ to the denom.}\\ \because\ \left(\frac{1}{2}\text{ co-eff. of } x\right)^2 = \frac{1}{4}\end{array}\right]$$

$$= \int \frac{1}{\sqrt{\left(\frac{1}{2}\right)^2 - \left(z - \frac{1}{2}\right)^2}}\ .\ dz$$

$$\left(z - \frac{1}{2}\right) = y \quad \Rightarrow \quad dz = dy$$

$$I = \int \frac{1}{\sqrt{(1/2)^2 - y^2}}\ .\ dy \qquad \left[\text{By using} \int \frac{1}{\sqrt{a^2 - x^2}}\ .\ dx = \sin^{-1}\left(\frac{x}{a}\right) + c\right]$$

$$= \sin^{-1}\left(\frac{y}{1/2}\right) + c$$

$$= \sin^{-1}\left(\frac{z - 1/2}{1/2}\right) + c \qquad [\because\ y = z - 1/2]$$

Solve another Integral

$$I = \int \sqrt{\frac{1+x}{x}} \, . \, dx$$

$$= \int \sqrt{\frac{1+x}{x} \times \frac{1+x}{1+x}} \, dx \qquad \text{[Multiply and divided by } (1 + x)]$$

$$= \int \sqrt{\frac{(1+x)^2}{x(1+x)}} \, . \, dx = \int \frac{1+x}{\sqrt{x+x^2}} \, . \, dx$$

Let us write :

$$1 + x = \lambda \, . \, \frac{d}{dx}(x + x^2) + \mu$$

$$\Rightarrow \qquad 1 + x = \lambda(1 + 2x) + \mu \qquad \text{...(1)}$$

$$\Rightarrow \qquad 1 + x = 2\lambda x + \lambda + \mu$$

Comparing the coefficients of x and the constant terms, we have

$$1 = 2\lambda \quad \Rightarrow \quad \lambda = \frac{1}{2}$$

and

$$1 = \lambda + \mu \quad \Rightarrow \quad \mu = 1 - \lambda = 1 - \frac{1}{2} = \frac{1}{2} \, .$$

Putting the values of λ and μ in (1),

$$1 + x = \frac{1}{2}(1 + 2x) + \frac{1}{2} \, .$$

$$\therefore \qquad I = \int \frac{\frac{1}{2}(1+2x) + \frac{1}{2}}{\sqrt{x+x^2}} \, . \, dx$$

$$= \frac{1}{2}\int \frac{1+2x}{\sqrt{x+x^2}} \, dx + \frac{1}{2}\int \frac{1}{\sqrt{x+x^2}} \, . \, dx$$

$$\Rightarrow \qquad I = \frac{1}{2} \, I_1 + \frac{1}{2} \, I_2 \qquad \text{...(2)}$$

Now $\qquad I_1 = \int \frac{1+2x}{\sqrt{x+x^2}} \, dx$

Put $\qquad x + x^2 = z \quad \Rightarrow \quad (1 + 2x) \, dx = dz$

$$\therefore \qquad I_1 = \int \frac{1}{\sqrt{z}} \, . \, dz = \int z^{-1/2} \, . \, dz = \frac{z^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c_1 = 2\sqrt{z} + c_1$$

$$= 2\sqrt{x + x^2} + c_1 \qquad \text{...(3)}$$

and $\qquad I_2 = \int \frac{1}{\sqrt{x+x^2}} \, . \, dx$

$$= \int \frac{1}{\sqrt{\left(x^2 + x + \frac{1}{4}\right) - \frac{1}{4}}} \, . \, dx \qquad \left[\begin{array}{l}\text{Add and subtract } \frac{1}{4} \text{ to the denom.} \\ \because \left(\frac{1}{2} \text{ co-eff. of } x\right)^2 = \frac{1}{4}\end{array}\right]$$

$$= \int \frac{1}{\sqrt{\left(x + \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2}} \, . \, dx$$

Put $\qquad x + \frac{1}{2} = z \quad \Rightarrow \quad dx = dz$

$$\therefore \quad I_2 = \int \frac{1}{\sqrt{z^2 - \left(\frac{1}{2}\right)^2}} \, . \, dz \qquad \left[\text{By using } \int \frac{1}{\sqrt{x^2 - a^2}} \, . \, dx = \log\left| x + \sqrt{x^2 - a^2} \right| + c\right]$$

$$= \log\left| z + \sqrt{z^2 - \left(\frac{1}{2}\right)^2} \right| + c_2 = \log\left| \left(x + \frac{1}{2}\right) + \sqrt{\left(x + \frac{1}{2}\right)^2 - \frac{1}{4}} \right| + c_2$$

$$= \log\left| \left(x + \frac{1}{2}\right) + \sqrt{x^2 + x} \right| + c_2 \qquad \ldots(4)$$

$\therefore$ **From equation (2),**

$$I = \frac{1}{2} I_1 + \frac{1}{2} I_2 \qquad \text{[Using (3) and (4)]}$$

Solve another problem

$$I = \int \frac{ax^3 + bx}{x^4 + c^2}\, dx = \int \frac{ax^3}{x^4 + c^2} . dx + \int \frac{bx}{x^4 + c^2} . dx$$

$$= a \int \frac{x^3}{x^4 + c^2} . dx + b \int \frac{x}{x^4 + c^2} . dx$$

$$\Rightarrow \quad I = a\, I_1 + b\, I_2 \qquad \text{...(1)}$$

Now $$I_1 = \int \frac{x^3}{x^4 + c^2} . dx$$

$$= \frac{1}{4} \int \frac{4x^3}{x^4 + c^2} . dx \qquad \text{[Multiply and divided by 4]}$$

$$= \frac{1}{4} \log \left| x^4 + c^2 \right| + c_1 \qquad \text{...(2)} \quad \left[\because \int \frac{f'(x)}{f(x)} . dx = \log |f(x)| + c \right]$$

and $$I_2 = \int \frac{x}{x^4 + c^2} . dx$$

$$= \frac{1}{2} \int \frac{2x}{(x^2)^2 + c^2}\, dx \qquad \text{[Multiply and divided by 2]}$$

Put $x^2 = z \;\Rightarrow\; 2x\, dx = dz$

$$= \frac{1}{2} \int \frac{1}{z^2 + c^2}\, dz \qquad \left[\text{By using} \int \frac{1}{x^2 + a^2} . dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + c \right]$$

Solve Integration root linear plus root linear in denominator

If $I = \int \frac{dx}{\sqrt{2x+3} + \sqrt{x+2}}$, then I equals

(a) $2(u - v) + \log\left|\frac{u-1}{u+1}\right| + \log\left|\frac{v-1}{v+1}\right| + C$

$u = \sqrt{2x+3}$, $v = \sqrt{x+2}$

(b) $\log\left|\frac{\sqrt{x+2} + \sqrt{2x+3}}{\sqrt{x+2} - \sqrt{2x+3}}\right| + C$

(c) $\log\left(\sqrt{x+2} + \sqrt{2x+3}\right) + C$

(d) is transcedental function in u and v, $u = \sqrt{2x+3}$

$v = \sqrt{x+2}$

Ans. (a), (d)

$$I = \int \frac{\sqrt{2x+3} - \sqrt{x+2}}{x+1}\,dx$$

$$= I_1 - I_2$$

where $I_1 = \int \frac{\sqrt{2x+3}}{x+1}\,dx$ and $I_2 = \int \frac{\sqrt{x+2}}{x+1}\,dx$

Put $2x + 3 = t^2$, in I_1, so that

$$I_1 = \int \frac{2t \cdot t}{t^2 - 1}\,dt = 2 \int \left[1 + \frac{1}{t^2 - 1}\right] dt$$

$$= 2\left[t + \frac{1}{2}\log\left|\frac{t-1}{t+1}\right|\right]$$

In I_2, put $x + 2 = y^2$, so that

$$I_2 = \int \frac{2y^2}{y^2 - 1}\,dy = 2y + \log\left|\frac{y-1}{y+1}\right|$$

Thus,

$$I = 2\left(\sqrt{2x+3} - \sqrt{x+2}\right) + \log\left|\frac{\sqrt{2x+3} - 1}{\sqrt{2x+3} + 1}\right|$$

$$+ \log\left|\frac{\sqrt{x+2} - 1}{\sqrt{x+2} + 1}\right| + C$$

Solve another Problem

Evaluate $\int \frac{\sin 2x\,dx}{(a+b\cos x)^2}$.

Solution:

We have $I = \int \frac{\sin 2x\,dx}{(a+b\cos x)^2} = 2\int \frac{\sin x \cos x\,dx}{(a+b\cos x)^2}$

Now put $a + b\cos x = t$

so that $-b\sin x\,dx = dt$.

Also $\cos x = \frac{(t-a)}{b}$.

$$\therefore\ I = -\frac{2}{b}\int \frac{(t-a)/b}{t^2}dt = -\frac{2}{b^2}\int\left[\frac{t}{t^2} - \frac{a}{t^2}\right]dt$$

$$= -\frac{2}{b^2}\int\left[\frac{1}{t} - \frac{a}{t^2}\right]dt = -\frac{2}{b^2}\left[\log t + \frac{a}{t}\right]$$

$$= -\frac{2}{b^2}\left[\log(a+b\cos x) + \frac{a}{a+b\cos x}\right].$$

A special Integral

$$\int \frac{(1-\sqrt{1+x+x^2})^2}{x^2\sqrt{(1+x+x^2)}}\,dx$$

Here we set $\sqrt{1+x+x^2} = xt + 1$, so that

$$x = \frac{2t-1}{1-t^2},\ dx = \frac{2t^2-2t+2}{(1-t^2)^2}\,dt \text{ and}$$

$$(1-\sqrt{1+x+x^2}) = \frac{-2t^2+t}{(1-t^2)}$$

Substitution of these values in the given integral transforms the problem in the form

$$\int \frac{(-2t^2+t)^2\,(1-t^2)^2\,(1-t^2)\,(2t^2-2t+2)}{(1-t^2)^2\,(2t-1)^2\,(t^2-t+1)\,(1-t^2)^2}\,dt$$

$$= +2\int \frac{t^2}{1-t^2}\,dt = -2t + \ln\left|\frac{1+t}{1-t}\right| + C$$

An advanced example

$$I=\int\frac{(x+1)}{x(1+xe^x)^2}\,dx$$

$$I=\int\frac{e^x(x+1)}{x\,e^x(1+xe^x)^2}\,dx$$

put $1 + xc^x = t,\ (xe^x + e^x)\ dx = dt$

$$I=\int\frac{dt}{(t-1)t^2}=\int\left(\frac{1}{1-t}+\frac{1}{t}+\frac{1}{t^2}\right)dt$$

$$=-\log|1-t|+\log|t|-\frac{1}{t}+C=\log\left|\frac{t}{1-t}\right|-\frac{1}{t}+C$$

$$=\log\left|\frac{1+xe^x}{-xe^x}\right|-\frac{1}{1+xe^x}+C=\log\left(\frac{1+xe^x}{xe^x}\right)-\frac{1}{1+xe^x}+C$$

Order and Power of a Differential Equation

Consider the given differential equation, $\sqrt{1+\left(\frac{dy}{dx}\right)^2}=\left(c\frac{d^2y}{dx^2}\right)^{\frac{1}{3}}$

Squaring on both the sides, we have

$$1+\left(\frac{dy}{dx}\right)^2=\left(c\frac{d^2y}{dx^2}\right)^{\frac{2}{3}}$$

Cubing on both the sides, we have

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^3=\left\{\left(c\frac{d^2y}{dx^2}\right)^{\frac{2}{3}}\right\}^3$$

$$\Rightarrow 1+\left(\frac{dy}{dx}\right)^6+3\left(\frac{dy}{dx}\right)^2+3\left(\frac{dy}{dx}\right)^4=c^2\left(\frac{d^2y}{dx^2}\right)^2$$

$$\Rightarrow c^2\left(\frac{d^2y}{dx^2}\right)^2-\left(\frac{dy}{dx}\right)^6-3\left(\frac{dy}{dx}\right)^4-3\left(\frac{dy}{dx}\right)^2-1=0$$

The highest order differential coefficient in this equation is $\frac{d^2y}{dx^2}$ *and its power is* 2.

Therefore, the given differential equation is a non – linear differential equation of second order and second degree.

Another Example

Consider the given differential equation,

$$\sqrt[3]{\frac{d^2y}{dx^2}} = \sqrt{\frac{dy}{dx}}$$

Cubing on both the sides of the above equation, we have

$$\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^{\frac{3}{2}}$$

Squaring on both the sides of the above equation, we have

$$\left(\frac{d^2y}{dx^2}\right)^2 = \left[\left(\frac{dy}{dx}\right)^{\frac{3}{2}}\right]^2$$

$$\Rightarrow \left(\frac{d^2y}{dx^2}\right)^2 = \left[\left(\frac{dy}{dx}\right)\right]^3$$

$$\Rightarrow \left(\frac{d^2y}{dx^2}\right)^2 - \left[\left(\frac{dy}{dx}\right)\right]^3 = 0$$

The highest order differential coefficient in this equation is $\frac{d^2y}{dx^2}$ *and its power is 2.*

Therefore, the given differential equation is a non – linear differential equation of second order and second degree.

We will see more examples at the end of the Chapter

Form the Differential equation by eliminating the unknown constants

$xy = ae^{x} + be^{-x} + c$

Sol : $xy = ae^{x} + be^{-x} + c$

$$\therefore x\frac{dy}{dx} + y = ae^{x} - be^{-x}$$

$$\therefore x\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{dy}{dx} = ae^{x} + be^{-x}$$

$$\therefore x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = xy - c$$

$$\therefore x\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} + 2\frac{d^2y}{dx^2} = x\frac{dy}{dx} + y$$

$$\therefore x\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} - x\frac{dy}{dx} - y = 0$$ is the differential equation.

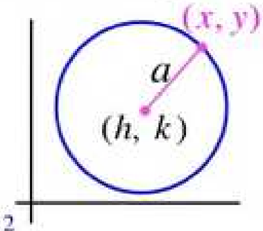
Another common example to find the Differential Equation

Find the differential equation of all circles of radious 'a' in a plane.

Sol: The family of circles of radious 'a' in a plane is

$$(x-h)^2+(y-k)^2=a^2 \longrightarrow (1)$$

where h, k are parameters.



$$\therefore \cancel{2}(x-h)+\cancel{2}(y-k)\frac{dy}{dx}=0 \longrightarrow (2)$$

$$\therefore 1+(y-k)\frac{d^2y}{dx^2}+\frac{dy}{dx}\cdot\frac{dy}{dx}=0 \Rightarrow y-k=-\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}} \longrightarrow (3)$$

$$\therefore \text{From (2), } x-h=-(y-k)\frac{dy}{dx}=\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}\frac{dy}{dx} \longrightarrow (4)$$

Substituting (3) and (4) in (1) we get

$$y-k=-\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}} \longrightarrow (3)$$

$$x-h=\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}\frac{dy}{dx} \longrightarrow (4)$$

$$(x-h)^2+(y-k)^2=a^2 \longrightarrow (1)$$

Substituting (3) and (4) in (1) we get

$$\left[\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}\frac{dy}{dx}\right]^2+\left[-\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}\right]^2=a^2.$$

$$\left[\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}\right]^2\left[\left(\frac{dy}{dx}\right)^2+1\right]=a^2.$$

$$\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^3}{\left(\frac{d^2y}{dx^2}\right)^2}=a^2$$ is the differential equation.

Now let us see Differential Equation types

1) Let us see an example of variable separable type

Solve $\frac{dy}{dx} = 1 + x + y + xy$

Sol: Given $\frac{dy}{dx} = 1 + x + y + xy$

$$= 1 + x + y(1+x)$$

$$= (1+x)(1+y)$$

$$\therefore \frac{dy}{1+y} = (1+x)dx$$

$$\therefore \int \frac{dy}{1+y} = \int (1+x)dx + c$$

$$\therefore \log(1+y) = x + \frac{x^2}{2} + c$$

Which is the required solution.

1.1 > Solve $v\frac{dv}{dx} + \frac{\mu}{x^2} = 0$

So $\int$ v dv + $\mu \int \frac{dx}{x^2}$ = C

1.2 > Solve

$$\left(1 + x^2\right) dy = \sqrt{y} \cdot dx$$

So $\frac{dx}{1+x^2}$ = $\frac{dy}{\sqrt{y}}$ Integrate both sides. You get

$$2\sqrt{y} - \tan^{-1} x = C$$

1.3 > Solve $y\,dx + x\,dy = 0$

Divide throughout by xy and we get

$\frac{dx}{x} + \frac{dy}{y} = 0$ so $\int \frac{dx}{x} + \int \frac{dy}{y} = C$

(So Multiplying factor or Integrating factor is 1 / xy)

So solution are

$\log xy = \log e^C$, i.e., $xy = e^C$; or $\log xy = \log C'$, i.e., $xy = C'$

1.4 > Solve

$(x - y^2)dx + 2xy\,dy = 0$

Will become variable separable after substitution

Substitute v = y^2 and then divide by x^2

will give $dx/x + d(v/x) = 0$

1.5 > Modifiable to variable separable by substitution

dy / dx = root (y-x)

Now as is it is not variable separable

Put y-x = u^2 so dy/dx - 1 = 2u (du / dx)

So 2u (du / dx) = u - 1

=> dx = (2u / (u -1)) du = 2 (1 + (1 / (u - 1)) du

=> x + c = 2 (u + ln |u-1|)

=> x + c = 2 (root (y-x) + ln |root(y-x) - 1 |

Reducible to Variable Separable

$$\frac{dy}{dx} = f(ax+by+c)$$

Let $ax+by+c=v$

Solve $\frac{dy}{dx} = [3x+y+4]^2$

Sol: Given $\frac{dy}{dx} = [3x+y+4]^2 \longrightarrow (1)$

Let $3x+y+4 = v \longrightarrow (2)$

$$\therefore 3+\frac{dy}{dx} = \frac{dv}{dx} \longrightarrow (3)$$

Eliminate y, by substituting (2),(3) in (1)

$$\frac{dv}{dx} - 3 = v^2$$

$$\therefore \frac{dv}{dx} = v^2+3$$

$$\therefore \frac{dv}{v^2+3} = dx$$

$$\therefore \int \frac{dv}{v^2+3} = \int dx + c$$

$$\therefore \frac{1}{\sqrt{3}} \tan^{-1} \frac{v}{\sqrt{3}} = x+c$$

The solution of the given d.e. is

$$\frac{1}{\sqrt{3}} \tan^{-1} \frac{3x+y+4}{\sqrt{3}} = x+c$$

2) Exact type

A differential equation written in the form

$M\,dx + N\,dy = 0$

is Exact if $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

2.1 > Solve $(x + 2y)\,x\,dx + (x^2 - y^2)\,dy = 0$

Observe $M = x\,(x + 2y)$ and $N = (x^2 - y^2)$

Also it is already in Mdx + N dy = 0 form (Else before testing M and N it has to brought to left)

$\partial M/\partial dy = 2x$ and $\partial N/\partial x = 2x$

So it Exact.

Solution is $\int$ M ∂x + $\int$ (Those terms of N without x) dy = C

So x^3 / 3 + 2y x^2 / 2 + ($-y^3$) / 3 = c

In M 2y is treated as constant and $-y^2$ is only taken out of N

Finally $\frac{1}{3}x^3 + x^2y - \frac{1}{3}y^3 = C$

2.2 > Solve $(xy^2 + x)\,dx + yx^2\,dy = 0$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}[xy^2 + x] = 2xy$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}[yx^2] = 2xy$$

So Solution is $\int (xy^2 + x)\ \partial x$ + $\int$ (0) dy = c (Note there are no terms in N without x)
=> $y^2 (x^2 /2)$ + ($x^2 / 2$) = c

3) Ways to convert an equation, which is not Exact type to Exact by Guessing the Multiplying factor

$y\,dx - x\,dy = 0$ is not Exact type

We can guess that multiplying by x^{-2}, $x^{-1}y^{-1}$, or y^{-2} Changes this to Exact. (So it is just an intelligent guess)

Other Guesses

Since $d\,(x^m y^n) = x^{m-1}y^{n-1}\,(my\,dx + nx\,dy)$

It has I.F of the form $x^{m-1}y^{n-1}$

$x^{km-1-\alpha}y^{kn-1-\beta}$ is an integrating Factor for any value of K

3.1 > Solve

$$y^3\,(y\,dx - 2x\,dy) + x^4\,(2y\,dx + x\,dy) = 0$$

As given it is not exact. But by multiplying throughout by

x^{-3} It becomes Exact

Check if the final answer is $2x^4y - y^4 = Cx^2$

3.2 > $(y^3 - 2yx^2)\,dx + (2xy^2 - x^3)\,dy$ = 0 seems to have xy as Integrating Factor

If the equation is of the form $f_1(x,\,y)\,y\,dx + f_2(x,\,y)\,x\,dy = 0$

Then 1 / $(Mx - Ny)$ is an Integrating factor

$$(1 + xy)\,y\,dx + (1 - xy)\,x\,dy = 0$$

has 1 / 2xy as the I.F

It is difficult to remember the following

If $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ is a pure function of x then $e^{\int f(x)dx}$

Or $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$ is a pure function of y then $e^{\int f(y)dy}$

4) Can we squeeze the differential coeffs ?

(Special Exact Differentials)

d(xy) = (dx)y + x (dy)

d(ln |xy|) = ((dx)y+ x(dy))/xy

$d(x^2+y^2) = 2\,x\,dx + 2\,y\,dy$

or $x\,dx + y\,dy = (½)d(x^2+y^2)$

d(x^2y^2) = 2x dx y^2 + x^2 2 y dy = 2xy (y dx + x dy)

So y dx + x dy = (1 / 2xy) d (x^2y^2)

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$$\frac{xdy - ydx}{x^2} = d\left(\frac{y}{x}\right)$$

$$\frac{xdy - ydx}{y^2} = -d\left(\frac{x}{y}\right)$$

$$\frac{xdy - ydx}{x^2 + y^2} = d\left(\tan^{-1}\left(\frac{y}{x}\right)\right)$$

$$\frac{xdx + ydy}{x^2 + y^2} = d\left(\frac{1}{2}\ln\left(x^2 + y^2\right)\right)$$

$$\frac{xdx + ydy}{\sqrt{x^2 + y^2}} = d\left(\sqrt{x^2 + y^2}\right)$$

$$\frac{xdx - ydy}{\sqrt{x^2 - y^2}} = d\left(\sqrt{x^2 - y^2}\right)$$

Don't forget

☺ dx + dy = d(x + y)

dx - dy = d(x - y)

4.1 > Solve

$$xdx + (y - \sqrt{x^2 + y^2})dy = 0$$

Reorganize as $\dfrac{xdx + ydy}{\sqrt{x^2 + y^2}} = dy$

$$d\left(\sqrt{x^2 + y^2}\right) = dy$$

So $\sqrt{x^2 + y^2} = y + c$

4.2 > Solve

$$\left(x^2 + y^2 + y\right)dx - xdy = 0$$

Reorganize to write as $dx + \dfrac{ydx - xdy}{x^2 + y^2} = 0$

=> dx = $d\left(\tan^{-1}\left(\dfrac{y}{x}\right)\right)$

So $x - \tan^{-1}\left(\dfrac{y}{x}\right) = c$

5) If powers of all terms are same in Numerator and Denominator then homogeneous. Put y= vx Differentiate and proceed

5.1 > Solve $\dfrac{dy}{dx}$ = (2x+3y) / (4x + 5y)

Put y = vx

$\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$ = (2x + 3vx) / (4x + 5vx) see x cancels out

=> $v + x\dfrac{dv}{dx}$ = (2 + 3v) / (4 + 5v) This variable separable type and gets solved easily

5.2 > Solve $\dfrac{dy}{dx} = \dfrac{x+y}{x}$

Put y = vx

$\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

So $v + x\dfrac{dv}{dx} = \dfrac{x+vx}{x} = 1 + v$

=> $x\frac{dv}{dx} = 1$

=> $\int dv = \int \frac{dx}{x}$

=> v = Ln x + c or ln x + ln c = ln (xc)

5.3 >

Solve $\frac{dy}{dx} = \frac{xy + x^2}{y^2}$

Put y = vx

=> $\frac{dy}{dx} = v + x\frac{dv}{dx}$ = (xvx + x^2) / (v^2 x^2) (x^2 cancels out)

$v + x\frac{dv}{dx}$ = (v + 1) / v^2

This is variable separable type

Let us discuss an example of modifiable to Homogeneous form

Solve $(2x+y+1)dx+(4x+2y-1)dy=0$

Sol: Given $(2x+y+1)dx+(4x+2y-1)dy=0$

$$\therefore \frac{dy}{dx}=-\frac{2x+y+1}{4x+2y-1}.\quad \text{Here } \frac{2}{4}=\frac{1}{2}$$

$$\therefore \frac{dy}{dx}=-\frac{2x+y+1}{2(2x+y)-1} \longrightarrow (1)$$

Let $2x+y=v \longrightarrow (2)$

sothat $2+\frac{dy}{dx}=\frac{dv}{dx} \longrightarrow (3)$

Eliminate y, by substituting (2),(3) in (1)

$$\frac{dv}{dx}-2=-\frac{v+1}{2v-1}$$

$$\therefore \frac{dv}{dx}=2-\frac{v+1}{2v-1}$$

$$\therefore \frac{dv}{dx}=2-\frac{v+1}{2v-1}=\frac{4v-2-v-1}{2v-1}=\frac{3v-3}{2v-1}$$

$$\therefore \frac{2v-1}{v-1}dv=3dx$$

$$\therefore 2v+\log|v-1|=3x+c$$

$\therefore$ The required solution is

$$2[2x+y]+\log|2x+y-1|=3x+c$$

i.e., $x+2y+\log|2x+y-1|=c$

Next example is more advanced example of modifiable to Homogeneous

5.4 >
Modifiable to Homogeneous
Solve $(2x^2 + 3 y^2 - 7)x\, dx - (3x^2 + 2 y^2 - 8) y\, dy = 0$
As is this is not homogeneous
But put $x^2 = u$ and $y^2 = v$

So $(2u + 3v - 7)\, du - (3u + 2v - 8)\, dv = 0$
By technique described below this can be Reduced or modified to homogeneous

Reducible to Homogeneous

$$(ax + by + c)\, dx + (a'x + b'y + c')\, dy = 0.$$

Put $v = x+h$ and $y = w + k$
Find h and k such that constants become zero

$h = \dfrac{b'c - bc'}{a'b - ab'}$ $\quad$ $k = \dfrac{ac' - a'c}{a'b - ab'}$

5.5 > Solve $(3y - 7x - 7)\, dx + (7y - 3x - 3)\, dy = 0$

Put $y = Y + h$ and $x = X + k$ search h and k such that constants are zero
$h = -1$ and $k = 0$

So $Y = v X$ will solve this

6) Linear

$$\frac{dy}{dx} + p(x)y = q(x)$$

The Integrating factor is $\exp(\int p\, dx)$

$$ye^{\int p dx} = \int Q e^{\int P dx} dx + C$$ where c is the constant of integration

Solve $\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}}-\frac{y}{\sqrt{x}}\right)\frac{dx}{dy}=1$

Sol: Given equation can be written as

$$\frac{e^{-2\sqrt{x}}}{\sqrt{x}}-\frac{y}{\sqrt{x}}=\frac{dy}{dx}$$

$$\therefore \frac{dy}{dx}+\frac{y}{\sqrt{x}}=\frac{e^{-2\sqrt{x}}}{\sqrt{x}}$$ which is linear.

Here $P=\frac{1}{\sqrt{x}}$, $Q=\frac{e^{-2\sqrt{x}}}{\sqrt{x}}$

$$\therefore \int P dx=\int\frac{1}{\sqrt{x}}dx=\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}=2\sqrt{x}$$

$$\therefore IF=e^{\int P dx}=e^{2\sqrt{x}}$$

$\therefore$ The solution is

$$y\ IF=\int Q\ IF\ dx+c$$

$$y\cdot e^{2\sqrt{x}}=\int\frac{e^{-2\sqrt{x}}}{\sqrt{x}}\cdot e^{2\sqrt{x}}dx+c$$

$$i.e.,\ y\cdot e^{2\sqrt{x}}=\int\frac{1}{\sqrt{x}}dx+c$$

$$i.e.,\ y\cdot e^{2\sqrt{x}}=2\sqrt{x}+c$$

6.1 > Solve $x\frac{dy}{dx} + y = 3x^2$

Convert to P(x) , Q(x) form

$\frac{dy}{dx} + \frac{1}{x}y = 3x$

=> $P = \frac{1}{x}, Q = 3x$

=> I.F = $\int P\,dx$ = $\int \frac{1}{x}$ dx = Ln x

$e^{\int Pdx} = e^{\log x} = x$

So solution is $y.e^{\int Pdx} = \int Q.e^{\int Pdx}.dx + C$

=> $y.x = \int 3x\,(x)dx + C$

=> $xy = \frac{3x^3}{3} + C$

=> $xy = x^3 + C$

6.2 >

dy/dx + (tan x) y = $\cos^2(x)$

I.F = $e^{\int \tan(x)dx} = e^{-\ln(\cos(x))} = e^{\ln(\sec(x))} = \sec(x)$

Multiply with Sec x throughout we get RHS as

$$\int \sec(x)\cos^2(x)dx = \int \cos(x)dx = \sin(x)$$

So Solution is $y = \frac{\sin(x) + C}{\sec(x)} = \left(\sin(x) + C\right)\cos(x)$

Modifiable to Linear

6.3 > $y\,y' \sin x = (\sin x - y^2) \cos x$

Put $y^2 = u \Rightarrow 2\,y\,y' = du / dx$
$((\sin x) / 2) (du / dx) = (\sin x - u) \cos x$
$\Rightarrow (du / dx) + 2u \cot x = 2 \cos x$
This is linear

So Integrating Factor is $\exp (2 \int \cot x \, dx)$
$= \exp (2 \ln \sin x) = \sin^2 x$

$\Rightarrow \sin^2 x (du / dx) + 2 u \sin x \cos x = 2 \cos x \sin^2 x$

$\Rightarrow d/dx$ of $(u \sin^2 x) = d / dx$ of $(2 \sin^3 x) / 3$

$\Rightarrow u \sin^2 x = (2/3) \sin^3 x + c$

$\Rightarrow y^2 = (2 / 3) \sin x + c \operatorname{Cosec}^2 x$

6.4 > Solve $dy / dx = (2xy) / (x^2 - 1 - 2y)$

Rewrite $2x (dx / dy) - (x^2 / y) = -(1 + 2y) / y$

Put $x^2 = u$ So $2x (dx / dy) = (du / dy)$

$\Rightarrow du / dy - (1 / y) u = - (1 + 2 y) / y$

This is linear in u and y

Integrating Factor exp(- $\int$ (1 / y) dy = exp (-ln |y|) = 1 / y

(1 / y) (du / dy) - u / y^2 = - (1 + 2y) / y^2

=> d/dy (u /y) = -1 / y^2 - 2/y

=> u / y = 1/y - 2 ln y + c

=> x^2 / y = 1 / y - 2 ln y + c

7) Bernoulli's Eqn $dy/dx + Py = Qy^n$

Divide by y^n and multiply by 1-n

We get $\frac{1-n}{y^n}\frac{dy}{dx} + (1-n)\,Py^{1-n} = (1-n)\,Q$

Put $y^{n-1} = v$ changes to dv/dx + (1-n)Pv = Q(1-n)

7.1 > Solve

$$dy/dx + x\sin 2y = x^3\cos^2 y$$

Divide by $\cos^2 y$ Put Tan y = v

Solve $x\frac{dy}{dx} + y = x^2 y^6$

Sol: Given equation can be written as

$\frac{dy}{dx} + \frac{1}{x}y = xy^6$, Which is Bernoulli's equation.

$$\therefore y^{-6}\frac{dy}{dx} + \frac{1}{x}y^{-5} = x \longrightarrow (1)$$

Let $y^{-5} = v \longrightarrow (2)$

sothat $-5y^{-6}\frac{dy}{dx} = \frac{dv}{dx} \longrightarrow (3)$

Eliminate y, by substituting (2),(3) in (1)

$$\frac{1}{-5}\frac{dv}{dx} + \frac{1}{x}v = x$$

$$\frac{1}{-5}\frac{dv}{dx} + \frac{1}{x}v = x$$

$\therefore \frac{dv}{dx} + \frac{-5}{x}v = -5x$ which is linear.

Here $P = \frac{-5}{x}$, $Q = -5x$

$$\therefore \int Pdx = \int \frac{-5}{x}dx = -5\log x = \log x^{-5}$$

$$\therefore IF = e^{\int Pdx} = e^{\log x^{-5}} = x^{5}$$

The solution is

$$v\ IF = \int Q\ IF\ dx + c$$

$$\therefore y^{5}\cdot x^{5} = \int -5x\ x^{5}dx + c$$

$$\therefore y^{-5}\cdot x^{-5} = -5\cdot\frac{x^{-4+1}}{-4+1} + c$$

i.e., $\frac{1}{y^5x^5} = \frac{5}{3x^3} + c$

Solve $\frac{dy}{dx} + x\sin 2y = x^3\cos^2 y$

Sol: Given equation can be written as

$$\frac{1}{\cos^2 y}\frac{dy}{dx} + x\frac{2\sin y\cos y}{\cos^2 y} = x^3$$

$$\therefore \sec^2 y\frac{dy}{dx} + 2x\tan y = x^3 \longrightarrow (1)$$

Let $\tan y = v \longrightarrow (2)$

so that $\sec^2 y\frac{dy}{dx} = \frac{dv}{dx} \longrightarrow (3)$

Eliminate y, by substituting (2),(3) in (1)

$\frac{dv}{dx} + 2xv = x^3$ which is linear.

Let $x^2 = t$ so that $2xdx = dt$

Here $P = 2x,\ Q = x^3$

$$\therefore \int Pdx = \int 2xdx = x^2$$

$$\therefore IF = e^{\int Pdx} = e^{x^2}$$

The solution is

$$v\ IF = \int Q\ IF\ dx + c$$

$$\therefore \tan y\ e^{x^2} = \int x^3 e^{x^2} dx + c$$

$$\therefore \int x^3 e^{x^2} dx = \int te^t \tfrac{1}{2} dt$$

$$= \frac{1}{2}\left[t\ e^t - 1\ e^t\right]$$

$$= \frac{1}{2}e^t\ t - 1$$

solution is

$$e^{x^2}\tan y = \frac{1}{2}e^{x^2}\ x^2 - 1\ + c$$

8) Factorise in D form

D is d by dx operator

8.1 >

Solve $(D^2 + 14D - 32)\, y = 0$

$D^2 + 14D - 32$ = 0 factorize and get D = 2 and -16

so dy / dx = 2 or dy/dx = -16

Unequal roots gives $y = C_1 e^{2x} + C_2 e^{-16x}$

If Equal roots then $(C_1 + C_2)\, e^{m_1 x}$ = y where m1 is the root

8.2> Solve $d^2y/dx^2 + dy/dx + y = 0$
=> (D^2 + D + 1)y = 0

Gives Imaginary roots so $y = e^{\alpha x}\,(A \cos \beta x + B \sin \beta x)$
Where root is α + iβ , α - iβ form

8.3 >
Solve $x \cdot d^2y/dx^2 + 2x \cdot dy/dx - 2y = 0$

Put $y = x^m$
We get $m\,(m-1) + 2\,(m-1) = 0$
$(m+2)\,(m-1) = 0$
m = -2 and m=1
So y = $C_1 x + C_2 x^{-2}$

To recall standard integrals

$f(x)$	$\int f(x)dx$	$f(x)$	$\int f(x)dx$
x^n	$\frac{x^{n+1}}{n+1} \quad (n \neq -1)$	$[g(x)]^n g'(x)$	$\frac{[g(x)]^{n+1}}{n+1} \quad (n \neq -1)$
$\frac{1}{x}$	$\ln\|x\|$	$\frac{g'(x)}{g(x)}$	$\ln\|g(x)\|$
e^x	e^x	a^x	$\frac{a^x}{\ln a} \quad (a > 0)$
$\sin x$	$-\cos x$	$\sinh x$	$\cosh x$
$\cos x$	$\sin x$	$\cosh x$	$\sinh x$
$\tan x$	$-\ln\|\cos x\|$	$\tanh x$	$\ln \cosh x$
$\operatorname{cosec} x$	$\ln\left\|\tan \frac{x}{2}\right\|$	$\operatorname{cosech} x$	$\ln\left\|\tanh \frac{x}{2}\right\|$
$\sec x$	$\ln\|\sec x + \tan x\|$	$\operatorname{sech} x$	$2\tan^{-1} e^x$
$\sec^2 x$	$\tan x$	$\operatorname{sech}^2 x$	$\tanh x$
$\cot x$	$\ln\|\sin x\|$	$\coth x$	$\ln\|\sinh x\|$
$\sin^2 x$	$\frac{x}{2} - \frac{\sin 2x}{4}$	$\sinh^2 x$	$\frac{\sinh 2x}{4} - \frac{x}{2}$
$\cos^2 x$	$\frac{x}{2} + \frac{\sin 2x}{4}$	$\cosh^2 x$	$\frac{\sinh 2x}{4} + \frac{x}{2}$

$f(x)$	$\int f(x)\, dx$	$f(x)$	$\int f(x)\, dx$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\tan^{-1}\frac{x}{a}$	$\frac{1}{a^2-x^2}$	$\frac{1}{2a}\ln\left\|\frac{a+x}{a-x}\right\| \quad (0<\|x\|<a)$
	$(a > 0)$	$\frac{1}{x^2-a^2}$	$\frac{1}{2a}\ln\left\|\frac{x-a}{x+a}\right\| \quad (\|x\| > a>0)$
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\frac{x}{a}$	$\frac{1}{\sqrt{a^2+x^2}}$	$\ln\left\|\frac{x+\sqrt{a^2+x^2}}{a}\right\| \quad (a > 0)$
	$(-a < x < a)$	$\frac{1}{\sqrt{x^2-a^2}}$	$\ln\left\|\frac{x+\sqrt{x^2-a^2}}{a}\right\| \quad (x>a>0)$
$\sqrt{a^2-x^2}$	$\frac{a^2}{2}\left[\sin^{-1}\left(\frac{x}{a}\right)\right.$	$\sqrt{a^2+x^2}$	$\frac{a^2}{2}\left[\sinh^{-1}\left(\frac{x}{a}\right) + \frac{x\sqrt{a^2+x^2}}{a^2}\right]$
	$\left.+\frac{x\sqrt{a^2-x^2}}{a^2}\right]$	$\sqrt{x^2-a^2}$	$\frac{a^2}{2}\left[-\cosh^{-1}\left(\frac{x}{a}\right) + \frac{x\sqrt{x^2-a^2}}{a^2}\right]$

Clairaut's differential equation

IIT JEE 1999

Solve

$$\left(\frac{dy}{dx}\right)^2 - x\left(\frac{dy}{dx}\right) + y = 0 \text{ is}$$

(a) $y = 2$ (b) $y = 2x$

(c) $y = 2x - 4$ (d) $y = 2x^2 - 4$

Solution—Given equation is of Clairaut's form

put $\frac{dy}{dx} = p$. Then equation is

$$y = px - p^2$$

Put $p = c$, we find

$$y = cx - c^2, \text{ where } c \text{ is any constant}$$

choose $c = 2$

$$y = 2x - 4$$

-

Clairaut's Equation :

The differential equation of the form

$$y = px + f(p);$$

is called Clairaut's equation. Its primitive is

$$y = C x + f(C)$$

and is obtained simply by replacing p by C in the given equation.

Example : Solve $p^4 - (x + 2y + 1) p^3 + (x + 2y + 2xy) p^2 - 2xyp = 0$

or $p(p - 1)(p - x)(p - 2y) = 0.$

The solutions of the component equations of first order and first degree.

Sol. : $\frac{dy}{dx} = 0$, $\frac{dy}{dx} = 1$, $\frac{dy}{dx} - x = 0$, $\frac{dy}{dx} - 2y = 0$

are respectively $y - C = 0$, $y - x - C = 0$, $2y - x^2 - C = 0$, $y - C e^{2x} = 0$,

The primitive of the given equation is

$$(y - C)(y - x - C)(2y - x^2 - C)(y - Ce^{2x}) = 0.$$

More Examples

Question 1

Given an Equation (dy/dx) (y + x – 1) – $(dy/dx)^2$ x = y

Take dy/dx = p

The equation is

$$y - px = \frac{p}{p-1}$$

or, $$y = px + \frac{p}{p-1}$$

It is of Clairaut's form, hence its solution is

$$y = cx + \frac{c}{c-1}$$

Question 2

Given an equation (dy/dx)^2 (x^2 - a^2) - 2xy(dy/dx) + y^2 - b^2 =0

Take dy/dx = p

So

The given equation is

$$p^2(x^2-a^2)-2pxy+y^2-b^2=0$$

or, $$y^2-2pxy+p^2x^2=a^2p^2+b^2$$

or, $$(y-px)^2=a^2p^2+b^2$$

or, $$y-px=\pm\sqrt{a^2p^2+b^2}$$

or, $$y=px\pm\sqrt{a^2p^2+b^2}$$

Both the component equations are of Clairaut's form

$\therefore$ The solution is

$$y=cx\pm\sqrt{a^2c^2+b^2}$$

or, $$(y-cx)^2=a^2c^2+b^2$$

Question 3

Given an equation (x – a) (dy / dx)^2 + (dy/dx) x = (1 + (dy / dx))y

The given equation is

$$(x-a)p^2 + px = (1+p)y$$

or, $$(1+p)y = px(p+1) - ap^2$$

or, $$y = px - \frac{ap^2}{p+1}$$

which is of Clairaut's form and hence its solutioin is

$$y = cx - \frac{ac^2}{c+1}$$

Question 4

Solve (dy/dx)^2 x^2 – 2 (dy/dx)^2 x + 2py – 2pxy – px + 2p + y^2 + y =0

So question is what is Clairaut's Differential Equations ?

The given equation is

$$p^2x^2 - 2p^2x + 2py - 2pxy - px + 2p + y^2 + y = 0$$

or, $$(y^2 - 2pxy + p^2x^2) + 2p(y - px) + (y - px) + 2p = 0$$

or, $$(y-px)^2 + (2p+1)(y-px) + 2p = 0$$

or, $$(y - px + 2p)(y - px + 1) = 0$$

Both the component equations are of Clairaut's form and hence the solution is

$$(y - cx + 2c)(y - cx + 1) = 0$$

Another Example

Solve $(1 + x^2)\frac{dy}{dx} + xy = x^3y^3$.

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Solution—$(1 + x^2)\frac{dy}{dx} + xy = x^3y^3$

$$\Rightarrow \quad \frac{1}{y^3}\frac{dy}{dx} + \frac{x}{1+x^2}\cdot\frac{1}{y^3} = \frac{x^3}{1+x^2}$$

Put $\quad \frac{1}{y^2} = v, \quad \frac{-2}{y^3}\frac{dy}{dx} = \frac{dv}{dx}$

$$\therefore \quad \frac{1}{-2}\frac{dv}{dx} + \frac{x}{1+x^2}v = \frac{x^3}{1+x^2}$$

$$\Rightarrow \quad \frac{dv}{dx} - \frac{2x}{1+x^2}v = \frac{-2x^3}{1+x^2}$$

Now I.F $= e^{\int \frac{-2x}{1+x^2} dx} = e^{-\log(1+x^2)} = \frac{1}{1+x^2}$

The solution is

$$v \cdot \frac{1}{(1+x^2)} = -2\int \frac{x^3}{(1+x^2)^2} dx + c$$

Put $\quad 1 + x^2 = t, \quad 2x dx = dt$

$$\therefore \quad \frac{v}{1+x^2} = -\int \frac{(t-1)}{t^2} dt + c$$

$$\frac{v}{1+x^2} = -\int \left[\frac{1}{t} - \frac{1}{t^2}\right] dt + c$$

$$= -\left[\log t + \frac{1}{t}\right] + c$$

$$\Rightarrow \quad \frac{v}{1+x^2} = -\left[\log(1+x^2) + \frac{1}{1+x^2}\right] c$$

Returning to y

$$\frac{1}{y^2(1+x^2)} = -\left[\log(1+x^2) + \frac{1}{1+x^2}\right] + c.$$

More examples teaches us better

The order of a differential equation is the order of the highest derivative included in the equation.

Find the order of the following Differential Equations

1. $\frac{dy}{dx} + y^2 x = 2x$
2. $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$
3. $10y'' - y = e^x$
4. $\frac{d^3y}{dx^3} - x\frac{dy}{dx} + (1-x)y = \sin\ y$

1. The highest derivative is dy/dx, the first derivative of y. The order is therefore 1

2. The highest derivative is d^2y / dx^2, a second derivative. The order is therefore 2

3. The highest derivative is the second derivative y". The order is 2

4. The highest derivative is the third derivative d^3 / dy^3. The order is 3

Another example

$$\frac{d^3x}{dt^3} + \frac{d^{2x}}{dt^2} + \left(\frac{dx}{dt}\right)^2 = e^t$$

The highest order differential coefficient is $\frac{d^3x}{dt^3}$ and its power is 1.

So, it is a non-linear differential equation with order 3 and degree 1.

Another example

$$\left(\frac{dy}{dx}\right)^2 + \frac{1}{\left(\frac{dy}{dx}\right)} = 2$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)^3 + 1 = 2\left(\frac{dy}{dx}\right)$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)^3 - 2\left(\frac{dy}{dx}\right) + 1 = 0$$

This is a polynomial in $\frac{dy}{dx}$.

The highest order differential coefficient is $\frac{dy}{dx}$ and its power is 3.

So, it is a non-linear differential equation with order 1 and degree 3.

Which of these differential equations are linear?

1. $\frac{dy}{dx} + x^2 y = x$
2. $\frac{1}{x}\frac{d^2y}{dx^2} - y^3 = 3x$
3. $\frac{dy}{dx} - \ln y = 0$
4. $\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = 2\sin x$

1. Both dy/dx and y are linear. The differential equation is linear

2. The term y^3 is not linear. The differential equation is not linear

3. The term ln y is not linear. This differential equation is not linear

4. The terms d^3y / dx^3, d^2y / dx^2 and dy / dx are all linear. The differential equation is linear

Determine the order and state the linearity of each differential below

1. $\left(\frac{d^3y}{dx^3}\right)^4 + 2\frac{dy}{dx} = \sin x$
2. $\frac{dy}{dx} - 2xy = x^2 - x$
3. $\frac{dy}{dx} - \sin y = -x$
4. $\frac{d^2y}{dx} = 2xy$

1. order 3 , non linear

2. order 1 , linear

3. order 1 , non linear

4. order 2 , linear

Example of a variable separable type

Solve $3e^x \tan y dx + (1+e^x)\sec^2 y dy = 0$ given $y = \frac{\pi}{4}$ when $x = 0$.

Sol: Given $3e^x \tan y dx + (1+e^x)\sec^2 y dy = 0$

Dividing with $\tan y \cdot (1+e^x)$, we get

$$\frac{3e^x}{1+e^x}dx + \frac{\sec^2 y}{\tan y}dy = 0$$

$$\therefore \int \frac{3e^x}{1+e^x}dx + \int \frac{\sec^2 y}{\tan y}dy = c$$

$$\therefore 3\log(1+e^x) + \log(\tan y) = \log C$$

$$\therefore (1+e^x)^3 \tan y = C$$

Put $x = 0$ and $y = \frac{\pi}{4}$

$$(1+e^0)^3 \tan\frac{\pi}{4} = C \Rightarrow C = 8$$

$\therefore$ The required solution is

$$(1+e^x)^3 \tan y = 8$$

Good Luck to you for your Preparations, References, and Exams

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